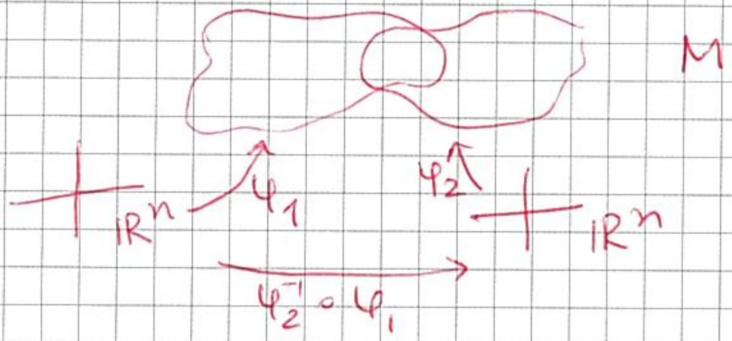
Lecture 12: Special topics

- plan for exams
- more solutions coming, please discuss on Discord!

Topic 1: Exotic $\mathbb{R}^4$s

Recall: Given a TOP mfld M^n, a sm. str. is a maximal atlas of charts on M s.t. the transition maps are smooth (on $\mathbb{R}^n$)



M smooth if $\varphi_2^{-1}\circ\varphi_1$ C^∞ for all such pairs.

A priori, a given TOP mfld may have 0, finite, or infinite sm. str.

Note: $\exists$ only countable compact TOP ($\Rightarrow$ sm) mflds

Note: The study of existence/uniqueness of sm str. is called smoothing theory

Remarks: S^7 has 28 distinct sm str. up to o.p. diffeo. [15 if allow o.r.] [Milnor, Kervaire–Milnor]

: High D (≥ 5) compact mflds admit at most finite sm. str.

4D compact mflds exist with inf. many sm. str, e.g. K3 surface $\{[x,y,z,w]\in\mathbb{C}P^3 \mid 0=x^4+y^4+z^4+w^4\}$

: Smooth 4D PC: $\exists$ nonstd sm str. on S^4? [open]

: $\exists$ non-smoothable manifolds of dim ≥ 4 but not dim ≤ 3

e.g. E8 manifold.

Theorem [Moise, Stallings] For $n\neq 4$, there is a unique sm. on $\mathbb{R}^n$, up to o.p. diffeomorphism.

[Taubes] $\exists$ uncountably many sm. str. on $\mathbb{R}^4$, up to o.p. diffeo.

[Gompf] $\exists$ construction w. input some knot $K\subseteq S^3$ which is TOP slice but not sm. slice and output an exotic sm. str. on $\mathbb{R}^4$.

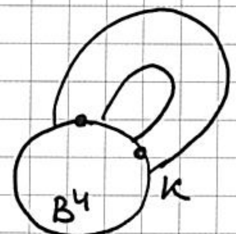
Note: A sm. str. on $\mathbb{R}^4$ which is not o.p. diffeo to the std sm. str. is called exotic.

[Gompf]: uncountably many exotic $\mathbb{R}^4$s arise this way.

Goal for Topic 1: this construction
Need following input:

Recall: $X_0(K) := B^4 \cup \underbrace{D^2 \times D^2}$ **0-trace**

0-framing of νK $\to \partial D^2 \times D^2$



Trace embedding lemma: Let $K \subseteq S^3$ a knot

K is sm/top slice $\iff$ $X_0(K)$ has a sm/top emb. in $\mathbb{R}^4_{std}$.

"collared" 0-dim analogue of loc. flat.

Pf: $(\Rightarrow)$

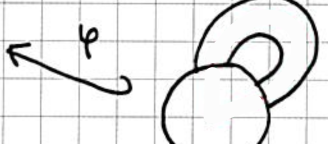
$\Big\}\ S^4 = \mathbb{R}^4_{std} \cup \{pt\}$

[why is it the 0-trace? vs n-trace?]

$(\Leftarrow)$

S^4

$X_0(K) \overset{\varphi}{\hookrightarrow} \mathbb{R}^4_{std} \hookrightarrow S^4$



Warning: Smooth 4D Schoenflies is still open!!

ie. given $\varphi: S^3 \overset{sm}{\hookrightarrow} S^4$

let C_1, C_2 denote the two components of $S^4 \setminus \varphi(S^3)$

Question: Is $\overline{A_i} \underset{\text{diffeo}}{\cong} B^4$?

[Analogue known in all other dimensions]

But [Palais] given $\varphi: B^4 \overset{sm}{\hookrightarrow} S^4$ then $\overline{S^4 \setminus \varphi(B^4)} \underset{\text{diffeo}}{\cong} B^4$.

So the $D^2 \times D^2$ gives a (tub. nbd) of a slice disc for K in this B^4. $\square$

Highly nontrivial input [Quinn] Let X^4 be a connected, noncompact 4-mfld. Then any sm str on ∂X extends a sm str on X.

Gompf's construction:

Let $K \subseteq S^3$ TOP slice but not sm. slice, e.g. $Wh^+(RHT)$.

Then $\exists\ \varphi : X_0(K) \xhookrightarrow{TOP} \mathbb{R}^4$.

We will now "construct" a $\overset{(new)}{\ }$ sm. str. on $\mathbb{R}^4$.

- $\varphi(X_0(K))$ is smooth

Check: $\mathbb{R}^4 \setminus Int(\varphi(X_0(K)))$ connected, non compact 4-mfd

$\Longrightarrow$ extend sm. str. on $\partial(\varphi(X_0(K)))$ to the comp.

Let $R := \mathbb{R}^4$, equipped w. above sm. str.

Then $R \underset{\text{diffeo}}{\not\cong} \mathbb{R}^4_{std}$, since if so, we would have $X_0(K) \xhookrightarrow{sm} \mathbb{R}^4_{std}$ which would $\Rightarrow\Leftarrow$ since K not sm slice

□

Topic 2: High D knots.

[see Lecture 1 for more overview]

In general a k-knot is an embedding $S^k \hookrightarrow S^n$. [in approp CATegory]

[unknotted := emb disc in S^n] codim := $n-k$.

Codim 1: Schoenflies problem [open in $\overset{PL}{\overset{\shortparallel}{\text{DIFF}}}$, $n=4$]

[Zeeman, Stallings] In PL, TOP, nontrivial knotting only in codim 2.
[except for 4D Schoenflies

[Haefliger, Levine] $\exists$ knotted smooth $S^{4k-1} \hookrightarrow S^{6k}$ e.g. $S^3 \hookrightarrow S^6$.

But that is about isotopy. What about concordance?
- by above, we focus on codim 2 case.

[Levine] $\mathcal{C}^{PL/TOP}_{S^{2k-1} \hookrightarrow S^{2k+1}}, k \geq 3 \equiv$ alg. concordance gp (analogue)

[for $\mathcal{C}_{S^3 \hookrightarrow S^5}$, there is an added $\mathbb{Z}/2$]

for $\mathcal{C}^{DIFF}_{S^{2k-1} \hookrightarrow S^{2k+1}}$, also get additional input from the gp of exotic $(2k-1)$-spheres

[Kervaire] All even dim knots are slice. ← sketch at end if time

Before doing any of this, an example! [Option 1: Glue together slice discs] (cart before horse)

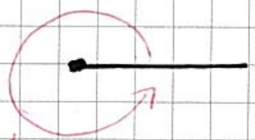
[Artin] Spun 2-knots.

Define $\mathbb{R}^3_+ := \{(x_1, x_2, x_3, 0) \mid x_3 \geq 0\} \subseteq \mathbb{R}^4$
with boundary $\mathbb{R}^2 := \{(x_1, x_2, 0, 0)\}$

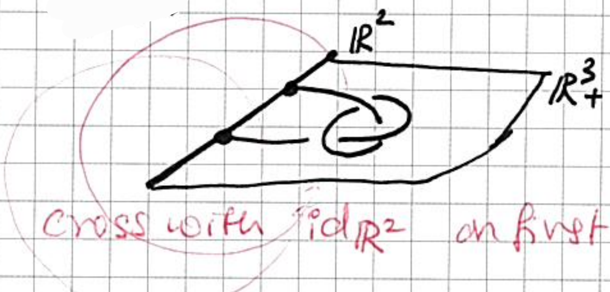
Then for $x := (x_1, x_2, x_3, 0) \in \mathbb{R}^3_+$, $0 \leq \theta \leq 2\pi$, let

$$x_\theta := (x_1, x_2, x_3 \cos\theta, x_3 \sin\theta)$$

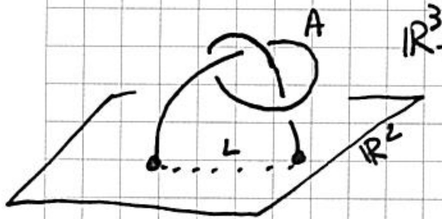
Given $X \subseteq \mathbb{R}^3_+$ we have $\mathrm{spin}(X) := \{x_\theta \mid x \in X,\ 0 \leq \theta \leq 2\pi\}$.



$\mathbb{R}^1_+$ spins to give $\mathbb{R}^2$, (around a pt)



Cross with $\mathrm{id}_{\mathbb{R}^2}$ on first two coords



e.g. spun trefoil

Let A: arc in $\mathbb{R}^3_+$, endpts on $\mathbb{R}^2$.

Then $\mathrm{spin}(A)$ is called a <u>spun 2-knot</u>

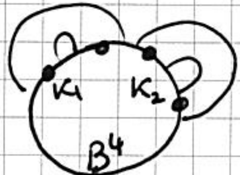
Similarly: - twist spinning
- roll spinning

<u>Proposition</u>: Suppose A: arc in $\mathbb{R}^3_+$, endpts on $\mathbb{R}^2$.
Let L: arc in $\mathbb{R}^2$, connecting endpts of A ↓ skipped this
↳ straight.

Then $\pi_1(\mathbb{R}^4 \setminus \mathrm{spin}(A)) \cong \pi_1(\mathbb{R}^3 \setminus A \cup L)$.

<u>Proof of Kervaire result</u> (sketch)

Key input: Any closed, orientable 3-mfld can be obtained ~~by~~:
[Milnor, Kaplan] as the boundary of:



attaching $D^2 \times D^2$s to the comps of a link $L \subseteq S^3$, s.t. each attached is by even framing.

[upgraded version of the statement that every 3-mfld is the result of Dehn surgery of a link]

Upside down: Every closed, oriented 3-mfld M can be changed to S^3 by attaching $D^2 \times D^2$s, attached along (even) framed circles in M.

Given $K: S^2 \hookrightarrow S^4$, let F be a Seifert 3-mfld, i.e. F is a compact oriented 3mfld $\subseteq S^4$ w. $\partial F = K$.

By the key input, we know $F \rightsquigarrow B^3$ by attaching $D^2\times D^2$'s along a link $L \subseteq F$.

The components of L bound disjoint embedded 2-discs in B^5 (since $5 \geq 2+2$, we can use transversality).

Framing work: the 2-discs have trivial normal bundles so the (even) framings on the boundary extend. $\square$

↑ skipped above

While it seems like conc. of ~~lower~~ even dim is not interesting, we still have:

(Open) question: Are all 2-links slice? 2k-links?

In general, there are also interesting questions about isotopy of 2-knots/links. E.g.

[Freedman] If $K: S^2 \hookrightarrow S^4$ sm/loc. flat has $\pi_1(S^4\setminus K) \cong \mathbb{Z}$ then K is (top) unknotted.

[Open] question: Is such a K sm. unknotted?

(Open) question: Does there exist $K: S^2 \hookrightarrow S^4$ loc. flat, which is not TOP isotopic to a sm 2-knot?

__Remark:__ One also considers $\Sigma_g \hookrightarrow S^4$ __surface knots__.